MAT 536 SPRING 2021 EXTRA CREDIT

1. Application of Hadamard formula

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R > 0$. Prove that if the series

$$\sum_{n=0}^{\infty} f^{(n)}(z)$$

converges for some $z_0 \in \mathbb{D}_R(0)$, then in fact $R = \infty$, the function f is entire and this series converges uniformly on every bounded subset of $\mathbb{C}$.

2. Let $f(z)$ be as in the previous problem. Show that if the sequence $\{\sqrt[n]{|f^{(n)}(z)|}\}$ is bounded at some point $z_0 \in \mathbb{D}_R(0)$, than again $R = \infty$, the function f is entire and the sequence is bounded at all z. Moreover, show that in this case $\limsup \sqrt[n]{|f^{(n)}(z)|}$ is the same for all z.

2. Behavior of power series on their circle of convergence

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 1$ and let $\mathbb{D}$ be an open unit disk with the center at the origin. A point ζ on the unit circle $S^1 = \partial\mathbb{D}$ is called a *regular point* for a holomorphic function f on $\mathbb{D}$, if there is an open disk $\mathbb{D}_r(z_0)$ of radius $r > 0$ with center z_0, $|z_0| < 1$, such that $\zeta \in \mathbb{D}_r(z_0)$ and corresponding power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges absolutely on $\mathbb{D}_r(z_0)$. In particular, $\lim_{z\to\zeta} f(z)$ from the inside exists. Otherwise, the point $\zeta \in S^1$ is called *singular*.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 1$ and $z_0 \in \mathbb{D}$. Prove that for $|z - z_0| < 1 - |z_0|$

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n, \quad \text{where} \quad b_n = \frac{f^{(n)}(z_0)}{n!}.$$

In particular, the series in the right-hand side is absolutely convergent.
(Hint: Use $z^n = (z - z_0 + z_0)^n$.)

2*. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 1$. Prove that $f(z) = (1 - z)\sum_{n=0}^{\infty} s_n z^n$, where $s_n = \sum_{k=0}^{n} a_k$.

3*. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 1$. Prove that if all coefficients a_n are real and the series $\sum_{n=0}^{\infty} a_n$ diverges to ∞ or $-\infty$, the $z = 1$ is a singular point.
(Hint: Use the previous problem.)

4*. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with real a_n and with the radius of convergence $R = 1$. Show by example that a weaker condition $|a_0 + \cdots + a_n| \to \infty$ does not guarantee that $z = 1$ is a singular point.

5*. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with with real a_n and with the radius of convergence $R = 1$. Prove that if coefficients $a_n \geq 0$ for all n, then $z = 1$ is a singular point (Pringsheim theorem).
(Hint: Use Problem **1**.)

6*. Prove that for the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ every point on the unit circle is singular.